Rouquier

Representation theory — structure (alg, group) acting on vector spaces (ab. groups, sets)

2-rep th. : monoidal category (additive) acting on category (additive, abelian, triang. ...)

(get 2 cat of such 2-rep.) all categories $\rightsquigarrow$ 2-category

3-rep. etc ...

Rep. th.

- Interesting Examples : complex semisimple Lie alg. Hecke alg.
- relation with top (3d TQFT, e.g. Reshetikhin-Turaev)
- construct interesting vector spaces (e.g. Fock spaces) as repr.

leads to combinatorial / arithmetical identities

2-Rep. th.

- "interesting" examples : in particular, one for each of ex. above
- relation with top 4d TQFT Crane-Frenkel
- construct abelian / triangular cat. of interest as rep.

? combinatorial property of gauge-theoretic invariants

? geometric Langlands

$\mathfrak{g}$: symmetrizable KM Lie alg.

$\rightarrow$ can construct a monoidal category $\mathcal{C}_{\mathfrak{g}}$

ex (Yesterday) $\mathfrak{g} = \mathfrak{sl}_2$

- E, F generating obj
- adjunction
- $X \in \operatorname{End} E$, $T \in \operatorname{End} E^2$ (q, f : parameter) $\neq 1$

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relations $X-a$: loc. nilp. (*)

$(T+1)(T-q)=0$

braiding with T

$T(1_E X)T = q(X 1_E)$

On K_0 $[E],[F]$ gives $\mathfrak{sl}_2(\mathbb{C})$ (*)

* $\mathfrak{g} = \mathfrak{sl}_{n+1}$ assume $q^i \neq 1$ $1 \leq i \leq n$

replace (*) by eigenvalue$(X) = \{a, qa, \cdots q^{n-1}a\}$

gives $E = E_0 \oplus \cdots \oplus E_{n-1}$ eigen sp. decomp.

$F =$ ..

(*) Ask $[E_i], [F_i]$ gives $\mathfrak{sl}_{n+1}(\mathbb{C})$

* $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ assume $q^n = 1$, $q^i \neq 1$ $1 \leq i \leq n-1$

eig.$(X) = \{a, qa \cdots, q^{n-1}a\}$

· $\mathfrak{g} = \mathfrak{sl}_\infty$ $q^n \neq 1$ $\forall n$ eig$(X) \in q^{\mathbb{Z}}$

<u>Conj</u>. (Thm 1, 2 extend from $\mathfrak{sl}_2$ to $\mathfrak{g}$)

$\mathcal{V} \supset \mathcal{C}_\mathfrak{g}$ $\Rightarrow$ 1) $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$

2) $0 = \mathcal{V}^{\leq -1} \subset \mathcal{V}^{\leq -2}$ $\qquad \mathcal{V}^{\leq i} / \mathcal{V}^{\leq i-1}$ isotypic

3) If $\mathbb{C} \otimes K_0(\mathcal{V})$ is a mult. of L irr.

then $\mathcal{V} \simeq \underset{\text{mult.}}{\mathcal{M}} \otimes \mathcal{V}(L)$ depending only on L "minimal cat"

$*\ \mathcal{V}$: abelian

$[S]$ canonical basis in K_0 $[$Proj. indec$]$: dual canonical basis

S : simple

($\mathcal{V}$: triangulated NO)

Conj : L simple, char 0 this is the canonical basis for $\mathcal{V} = \mathcal{V}(L)$)

$\mathrm{hd}(E_i S)$ 0 or simple $\longrightarrow$ Kashiwara operator

$\mathcal{V} = \mathcal{V}(L)$

crystal limit

Examples

$\mathcal{V} = \bigoplus_{n \geq 0} H_n^f(q)\text{-mod}$ (over $\mathbb{C}$) $q = \sqrt[p]{1}$ $p > 1$

$\mathcal{C}_{\widehat{sl}_p}$ action $\left(\begin{array}{l} E, F \ : \mathrm{Ind}, \mathrm{Res} \\ X \ : \text{image of } X_{n+1} \in H_{n+1} \\ T : T_n \end{array} \right.$ H_n^f : quot. of H_n

eigen p-th root of 1

canonical basis : Ariki's thm

$\left(\bigoplus_{n \geq 0} \mathbb{F}_p[S_n]\text{-mod} \curvearrowleft \mathcal{C}_{\widehat{sl}_p} \right.$ basis of simple $\neq$ cano. basis

$\mathcal{V} \overset{?}{=} \mathcal{V}(L)$ for some irr. L

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* $\mathfrak{g} = \mathfrak{gl}(V)$ $V = \mathbb{C}^n$

$E = V \otimes -$ on $\mathfrak{g}$-mod

$N \in \mathfrak{g}$-mod $\quad \mathfrak{g} \otimes N \to N \quad \rightsquigarrow \quad V \otimes N \xrightarrow{X(N)} V \otimes N$

$\mathfrak{g} = V \otimes V^*$ (Casimir)

$T = V \otimes V$

(Arakawa-Suzuki)

$U_q(\mathfrak{g})$-version でできる. $(q \neq 1)$

$\mathcal{C}_{\Pi \text{Sl}_\infty} \curvearrowright \mathfrak{g}$-mod $\quad$ g-m

*? L: irred. fin.dim. $\mathcal{U}(L)$ should ~~arise~~ arise as sheaves on Nakajima quiver varieties

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Khovanov

<u>Rem</u> $W = S_n \rightsquigarrow$ get categorification of B_n $\mathcal{C}_{B_n}$ $\rightsquigarrow$ triply graded link homology

<u>Conj</u> $\mathcal{C}_\mathfrak{g}$-rep. has a tensor product making it a "braid monoidal 2-category" giving a 4d TQFT whose de-categorification is Reshetikhin–Turaev 3d

No ~~No~~ coproduct?

Schur – Weyl type duality with above example

(work with A_∞-categories)

$\otimes$ is well-defined up to homotopy

Ex. $\mathfrak{g} = \mathfrak{sl}_2$

$$\mathcal{U}(\mathbb{C}^2) = \begin{pmatrix} \mathbb{C}\text{-mod} \\ \oplus \\ \mathbb{C}\text{-mod} \end{pmatrix}$$

$$\mathcal{U}(\mathbb{C}^2) \otimes \mathcal{U}(\mathbb{C}^2) = \begin{pmatrix} \mathbb{C}\text{-mod} \\ \oplus \\ \text{pr. block cat. } \mathcal{O} \\ \text{of } \mathfrak{sl}_2 \\ \oplus \\ \mathbb{C}\text{-mod} \end{pmatrix}$$

Only result : $\mathcal{U}(\mathbb{C}^2) \otimes -$: understand

$\mathcal{U}(\mathbb{C}^2) \otimes -$ and $- \otimes \mathcal{U}(\mathbb{C}^2)$ are different

Diff/alg geometry : construct moduli space, get invariants from them

(gauge theory)

cat. sheaves

cat. sheaves

?

How to construct category from other categories

NB